Wilberforce Pendulum modelling

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Introduction

This project requires the creation and investigation of a complex mechanical system as least as complex as a double pendulum. For this task, we have chosen the Wilberforce pendulum which was first described by the physicist Lionel Robert Wilberforce in 1896. The Wilberforce pendulum is constructed by hanging a mass which is free to rotate in the direction of the coils, with certain harmonic properties which allow for energy to alternate between purely vertical and rotational motion. This is achieved through a coupling of the rotational and spring potential energy of the system. To extend this concept we have placed the system on a free support and allowed for a sideways swinging motion. To our knowledge, this setup has never been modelled before.

This project focuses on the description and analysis of the pendulum, using the Euler-Lagrange equations of motion to predict the pendulum's path and subsequently compare numerical simulations of the system with experimental data. This document primarily serves to describe the theoretical implications of design choices for the pendulum and to predict its path using computational software.

The Wilberforce Pendulum With Free Support

The mass is free to slide along a frictionless rail. The Wilberforce pendulum consists of a cylindrical mass (length and moment of inertia ) suspended at its mid-point by a helical spring of natural length , stretched length l + s(t), longitudinal spring constant and torsional spring constant . We assume that the spring has negligible mass and is rigid along its axis throughout the experiment. The cylinder must hang such that its axis is always in the plane orthogonal to the direction of the spring, and is able to rotate along this plane. The spring makes an angle from the vertical and the tracked point on the cylinder creates an angle measured from its starting point in the rotational plane: see Figure 2 . The horizontal position of the mass is noted as from the origin : see Figure 1. We choose the reference of the gravitational potential energy to be zero at the height of the origin. Note that the motion is actually in 3 dimensions but the direction into the page is essentially irrelevant to our defined equations of motion, as it can be represented by .

Note that there are four coordinates, that is, θ, , and . Thus there will be four Euler-Lagrange equations which must be solved simultaneously.

Deriving the Euler-Lagrange equations

We start by deriving the kinetic and potential energies, and hence the Lagrangian.

Kinetic energy

As the entire system is free to slide along a rail, the whole system can obtain a common velocity and thus kinetic energy. Defining to be the velocity of the system in the direction, we have . Furthermore, the cylinder will have velocity relative to the system and rotational energy dependant on the rotational velocity given by and .

Hence, the total kinetic energy of the system is , so

Potential energy

We know that a spring with spring constant and stretch length has a spring potential . We also know that the torsional potential of the spring is . Since these potentials are coupled, we furthermore have the potential which represents the shared energy held during the transfer of energy between and in the system. As there are only two coupled components of potential, this relationship is linear and proportional to and . Thus, assume where is some constant. Finally, as this Wilberforce experiment has been made unique by the inclusion of traditional swinging pendulum motion, we also have gravitational potential .

Thus, the total kinetic energy of the system is , so